SOME REMARKS ON THE SIZE OF TUBULAR NEIGHBORHOODS IN CONTACT TOPOLOGY AND FILLABILITY

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Abstract. The well-known tubular neighborhood theorem for contact submanifolds states that a small enough neighborhood of such a submanifold N is uniquely determined by the contact structure on N, and the conformally symplectic structure of the normal bundle. In particular, if the submanifold N has trivial normal bundle then its tubular neighborhood will be contactomorphic to a neighborhood of $N \times \{\mathbf{0}\}$ in the model space $N \times \mathbb{R}^{2k}$.

In this article we make the observation that if (N, ξ_N) is a 3–dimensional overtwisted submanifold with trivial normal bundle in (M, ξ), and if its model neighborhood is sufficiently large, then (M, ξ) does not admit an exact symplectic filling.

In symplectic geometry many invariants are known that measure in some way the "size" of a symplectic manifold. The most obvious one is the total volume, but this is usually discarded, because one can change the volume (in case it is finite) by rescaling the symplectic form without changing any other fudamental property of the manifold. The first non-trivial example of an invariant based on size is the symplectic capacity [Gro85]. It relies on the fact that the size of a symplectic ball that can be embedded into a symplectic manifold does not only depend on its total volume but also on the volume of its intersection with the symplectic 2–planes.

Contact geometry does not give a direct generalization of these invariants. The main difficulties stem from the fact that one is only interested in the contact structure, and not in the contact form, so that the total volume is not defined, and to make matters worse the whole Euclidean space $\mathbb{R}^{2n+1}$ with the standard structure can be compressed by a contactomorphism into an arbitrarily small open ball in $\mathbb{R}^{2n+1}$.

A more successful approach consists in studying the size of the neighborhood of submanifolds. This can be considered to be a generalization of the initial idea since contact balls are just neighborhoods of points. In the literature this idea has been pursued by looking at the tubular neighborhoods of circles. Let (N, α_N) be a closed contact manifold. The product $N \times \mathbb{R}^{2k}$ carries a contact structure given as the kernel of the form $\alpha_N + \sum_{j=1}^{k}(x_j\, dy_j - y_j\, dx_j)$, where $(x_1, \ldots, x_k, y_1, \ldots, y_k)$ are the coordinates of the Euclidean space. If (N, α_N) is a contact submanifold of a manifold (M, α) that has trivial (as conformal symplectic) normal bundle, then one knows by the tubular neighborhood theorem that N has a small neighborhood in M that is contactomorphic to a small neighborhood of $N \times \{\mathbf{0}\}$ in the product space $N \times \mathbb{R}^{2k}$.

The contact structure on a solid torus V in $\mathbb{S}^1 \times \mathbb{R}^2$ depends in an intricate way on the radius of V [Eli91]. Later, examples of transverse knots in 3–manifolds were found whose maximal neighborhood is only contactomorphic to a small disk bundle in $\mathbb{S}^1 \times \mathbb{R}^2$ [EH05]. This is proved by measuring the slope of the characteristic foliation on the boundary of cylinders.

A different approach has been taken in [EKP06]. There it has been shown that a solid torus around $\mathbb{S}^1 \times \{\mathbf{0}\}$ in $\mathbb{S}^1 \times \mathbb{R}^{2k}$ of radius R cannot be "squeezed" into a solid torus of radius r, if $k \geq 2$, $R > 2$ and $r < 2$. However note that squeezing in the context of [EKP06] is different from the expected definition, and refers to the question of whether one subset of a contact manifold can be deformed by a global isotopy into another one.

The observation on which the present article is based is that sufficiently large neighborhoods of $N \times \{\mathbf{0}\}$ in $N \times \mathbb{R}^{2k}$ contain a generalized plastikstufe (for a definition of the GPS see Section 3), if N is an overtwisted 3–manifold. The construction of a GPS in a tubular neighborhood is explained in Section 4. In Section 5, we show that the existence of a GPS implies nonfillability, and so it follows in particular that an overtwisted contact manifold that is embedded into a fillable manifold cannot have a "large" neighborhood.

Unfortunately, the definition of "large" is rather subtle and does not lead to a numerical invariant, because such an invariant would depend on the contact form on the submanifold. One could simply multiply any contact form $\alpha_N + \sum_{j=1}^k (x_j\, dy_j - y_j\, dx_j)$ by a constant $\lambda > 0$, and then rescale the radii in the plane by a transformation $r_j \mapsto r_j/\sqrt{\lambda}$ to change the numerical invariant.

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1. Examples

First we give an easy example that shows that embedding an overtwisted 3–manifold into a fillable contact manifold does not pose a fundamental problem in positive codimension.

Example 1. Let M be an arbitrary orientable closed 3–manifold. Its unit cotangent bundle $\mathbb{S}(T^*M) \cong M \times \mathbb{S}^2$ has a contact structure defined by the canonical 1–form. The cotangent bundle T^*M together with the form $d\lambda_{\mathrm{can}}$ is an exact symplectic filling (and in fact, it can even be turned into a Stein filling).

Any contact manifold (M, α) can be embedded into $\big(\mathbb{S}(T^*M), \lambda_{\mathrm{can}}\big)$ just by normalizing α so that $\|\alpha\| = 1$. This defines a section in $\sigma:\, M \to \mathbb{S}(T^*M)$ with $\sigma^*\lambda_{\mathrm{can}} = \alpha$. This means that every (and in particular also every overtwisted overtwisted one) contact 3–manifold can be embedded into a Stein fillable contact 5–manifold.

Embedding a contact 3–manifold into a contact manifold of dimension 7 or higher restricts by using the h–principle and a general position argument to a purely topological question.

Our second and third example show that contact submanifolds can have infinitely large tubular neighborhoods.

Example 2. Let (M, α) be an arbitrary contact manifold, and let $(\mathbb{S}^{2n-1}, \xi_0)$ be the standard sphere. If $\dim M \geq 2n - 1$, then it is easy to give a contact embedding

$$\Big(\mathbb{S}^{2n-1} \times \mathbb{R}^{2k}, \alpha_0 + \sum_{j=1}^k (x_j\, dy_j - y_j\, dx_j)\Big) \hookrightarrow (M, \alpha)\ .$$

The proof works in two steps. For the embedding

$$\Big(\mathbb{S}^{2n-1} \times \mathbb{R}^{2k}, \alpha_0 + \sum_{j=1}^k (x_j\, dy_j - y_j\, dx_j)\Big) \hookrightarrow \big(\mathbb{S}^{2n+2k-1}, \alpha_0\big)$$

simply use the map $(z_1, \dots, z_n; x_1, y_1, \dots, x_k, y_k) \mapsto \frac{1}{\sqrt{1 + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}}\, (z_1, \dots, z_n, x_1 + iy_1, \dots, x_k + iy_k)$. Since $(\mathbb{S}^{2N-1}, \alpha_0)$ with one point removed is contactomorphic to $\mathbb{R}^{2N-1}$ with standard contact structure (see for example [Gei06, Proposition 2.13]) and since it is possible to embed the whole $\mathbb{R}^{2N-1}$ into an arbitrary small Darboux chart (see for example [CvS08, Proposition 3.1]), it follows that a general (M, α) contains embeddings of $\big(\mathbb{S}^{2n-1} \times \mathbb{R}^{2k}, \alpha_0 + \sum_{j=1}^k (x_j\, dy_j - y_j\, dx_j)\big)$.

Example 3. A generalization is obtained by choosing a contact manifold (N, α_N) that has an exact symplectic filling $(W, \omega = d\lambda)$. The Lioville field X_L is globally defined (see Section 2.1), and we can use its negative flow for finding an embedding of the lower half of the symplectization $(-\infty, 0] \times N$ where λ pulls back to $e^t\, \alpha_N$. The manifold $\mathbb{S}^1 \times W$ is together with the 1–form $d\vartheta + \lambda$ a contact manifold.

The standard model $(N \times \mathbb{R}^2, \alpha_N + r^2\, d\varphi)$ can be glued outside the 0–section onto $\mathbb{S}^1 \times W$, and this construction yields a closed contact manifold that contains the embedding of $N \times \mathbb{R}^2$. This example can be seen as an open book with binding N, page W, and trivial monodromy.

Not much is known about the different contact structures on $\mathbb{R}^{2n+1}$ for $n \geq 2$. There exists the standard contact structure ξ_0, and many different constructions to produce structures that are not isomorphic to the standard one (for example [BP90, Mul90, Nie06]). Unfortunately we do not have the techniques to decide whether these exotic contact structures are different from each other. A contact structure ξ on $\mathbb{R}^{2n+1}$ is called **standard at infinity** [Eli93], if there exists a compact subset K of $\mathbb{R}^{2n+1}$ such that $\bigl(\mathbb{R}^{2n+1} - K, \xi\bigr)$ is contactomorphic to $\bigl(\mathbb{R}^{2n+1} - \mathbb{D}_R, \xi_0\bigr)$ for a closed disk of an arbitrary radius R. A contact structure ξ on $\mathbb{R}^{2n+1}$ only admits a one-point compactificaton to a contact structure on the sphere, if ξ is standard at infinity. For most exotic contact structures it is not known whether they are standard at infinity or not. The only exception known to us so far was given in [KN07], where by removing one point from the sphere, we obtained an exotic contact structure ξ_{PS} on $\mathbb{R}^{2n+1}$ that *is* standard at infinity (but see also Example 5). A rather degenerate way of producing a contact structure that is not standard at infinity consists in taking the standard structure on $\mathbb{R}^{2n+1}$, and do the connected sum at every point $(0, \dots, 0, k) \in \mathbb{R}^{2n+1}$ with $k \in \mathbb{Z}$ with the sphere $(\mathbb{S}^{2n+1}, \xi_{PS})$.

Corollary 5 below yields a very explicit way to construct an exotic contact structure that is not standard at infinity.

Example 4. The contact manifold

$$\Bigl(\mathbb{R}^3 \times \mathbb{C}^k, \alpha_- + \sum_{j=1}^k r_j^2 \, d\vartheta_j\Bigr) ,$$

where (r_j, ϑ_j) are polar coordinates on the j–th factor of $\mathbb{C}^k$, does not embed into the standard sphere, and is hence not contactomorphic to the standard contact structure on $\mathbb{R}^{2k+3}$. Let $K \subset \mathbb{R}^3 \times \mathbb{C}^k$ be an arbitrary compact subset. By the same argument, it is easy to see that $\bigl(\mathbb{R}^3 \times \mathbb{C}^k - K, \alpha_- + \sum_j r_j^2 \, d\vartheta_j\bigr)$ still contains a GPS, so in particular it cannot be embedded into a "punctured" set $U - \{p\} \subset \bigl(\mathbb{R}^{2k+3}, \alpha_0\bigr)$ with the standard contact structure. It follows that $\bigl(\mathbb{R}^3 \times \mathbb{C}^k, \alpha_- + \sum_j r_j^2 \, d\vartheta_j\bigr)$ is "non standard at infinity".

Let (M, α) be a closed contact manifold that contains a contact submanifold N of codimension 2 with trivial normal bundle. A k–fold **contact branched covering** over M consists of a closed manifold $\widetilde{M}$, and a smooth surjective map $f : \widetilde{M} \to M$ such that the map f is a smooth k–fold covering over $M - N$, and there is an open neighborhood $\widetilde{U} \subset \widetilde{M}$ of $f^{-1}(N)$ diffeomorphic to $N \times \mathbb{D}_\varepsilon$, and a neighborhood $U \subset M$ of N diffeomorphic to $N \times \mathbb{D}_{\varepsilon^k}$ such that the map f takes the form

$$f : \, N \times \mathbb{D}_\varepsilon \to N \times \mathbb{D}_{\varepsilon^k}, \, (p, z) \mapsto (p, z^k) ,$$

when restricted to $\widetilde{U}$ (see [Gon87]).

Using the branched covering, it is easy to define a contact structure on $\widetilde{M}$. First isotope α in such a way that it becomes $\alpha|_{TN} + r^2 \, d\varphi$ on a subset $N \times \mathbb{D}_\delta \subset N \times \mathbb{D}_{\varepsilon^k}$ for some $\delta > 0$. The pull-back $\widetilde{\alpha} := f^*\alpha$ defines on $\widetilde{M}$ a 1–form that satisfies away from $f^{-1}(N)$ everywhere the contact property. Over the branching locus $f^{-1}(N)$, there is a subset $N \times \mathbb{D}_{\sqrt[k]{\delta}}$ in $\widetilde{U}$ where $\widetilde{\alpha}$ evaluates to $\alpha|_{TN} + kr^{2k} \, d\varphi$.

Remove the fiber $N \times \{0\}$ from $\widetilde{U}$ and glue in $N \times \mathbb{D}_\delta$ via the map $F : \, (p, re^{i\varphi}) \mapsto (p, \sqrt[k]{r} \, e^{i\varphi})$ along $N \times \bigl(\mathbb{D}_{\sqrt[k]{\delta}} - \{0\}\bigr)$. The pull-back $F^*\widetilde{\alpha}$ yields $\alpha|_{TN} + kr^2 \, d\varphi$ on the punctured disk bundle, which we can easily extend to the whole patch we are gluing in. We denote this slightly modified contact form again by $\widetilde{\alpha}$. By using a linear stretch map on the disk, we finally obtain that the submanifold $f^{-1}(N) \cong N$ has with respect to the model form

$$\bigl(N \times \mathbb{R}^2, \, \alpha|_{TN} + r^2 \, d\varphi\bigr)$$

a neighborhood inside $(\widetilde{M}, \widetilde{\alpha})$ that is at least of size $\sqrt{k} \, \delta$.

Example 5. There is an interesting contact structure on the odd dimensional spheres $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ given as the kernel of the 1–forms

$$\alpha_- = i \sum_{j=1}^{n} \bigl(z_j \, d\bar z_j - \bar z_j \, dz_j\bigr) - i \left(f \, d\bar f - \bar f \, df\right)$$

with $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$. This form is compatible with the open book with binding $B = f^{-1}(0)$, and fibration map $\vartheta = \bar f / |f|$. In abstract terms, this is the open book with page $P \cong T^*\mathbb{S}^{n-1}$ and monodromy map corresponding to the negative Dehn-Seidel twist.

An interesting feature of these spheres is that they can be stacked into each other via the natural inclusions $\mathbb{S}^3 \hookrightarrow \mathbb{S}^5 \hookrightarrow \mathbb{S}^7 \hookrightarrow \cdots$ respecting the contact form, and that $(\mathbb{S}^3, \alpha_-)$ is overtwisted.

We find a contact branched cover $f : \mathbb{S}^5 \to (\mathbb{S}^5, \alpha_-)$ given by $f(z_1, z_2, z_3) = \frac{(z_1, z_2, z_3^k)}{\|(z_1, z_2, z_3^k)\|}$ that is branched along $\mathbb{S}^3$. By choosing k large enough, we will obtain with the construction described above a contact structure on $\mathbb{S}^5$ that contains an embedding of $(\mathbb{S}^3, \alpha_-)$ with an arbitrary large neighborhood. According to Corollary 5 and Theorem 4, this contact structure will not admit an exact symplectic filling.

This result is unsatisfactory, since we do not get an explicit value for k. In fact, we expect that $(\mathbb{S}^3, \alpha_-)$ already has a large neighborhood in any of the spheres $(\mathbb{S}^{2n-1}, \alpha_-)$ so that taking $k = 1$ (that means not taking any branched covering at all) should already be sufficient.

2. Preliminaries

2.1. **Fillability.** In this section, we will briefly present some standard definitions and properties regarding fillability and J–holomorphic curves.

Definition. A **Liouville field** X_L is a vector field on a symplectic manifold (W, ω) for which

$$\mathcal{L}_{X_L} \omega = \omega$$

holds.

If (W, ω) is a symplectic manifold with boundary $M := \partial W$, and if X_L is a Liouville field on W that is transverse to M, then the kernel of the 1–form

$$\alpha := \omega(X_L, -)|_{TM}$$

defines a contact structure on M.

Definition. Let (M, ξ) be a closed contact manifold. A compact symplectic manifold (W, ω) with boundary $\partial W = M$ is called a **strong (symplectic) filling** of (M, ξ), if there exists a Liouville field X_L in a neighborhood of the boundary M pointing *outwards* through M such that X_L defines a contact form for ξ. If the vector field X_L is defined globally on W, we speak of an **exact symplectic filling**.

Remark 1. In a symplectic filling, we can always find a neighborhood of M that is of the form $(-\varepsilon, 0] \times M$ by using the negative flow of X_L to define

$$(-\varepsilon, 0] \times M \to W, \; (p, t) \mapsto \Phi_t(p) \; .$$

Denote the hypersurfaces $\{t\} \times M$ by M_t, and the 1–form $\omega(X_L, -)$ by $\widehat\alpha$. It is clear that $\widehat\alpha$ defines on every hypersurface M_t a contact structure. The Reeb field X_{Reeb} is the unique vector field on $(-\varepsilon, 0] \times M$ that is tangent to the hypersurfaces M_t, and satisfies both $\omega(X_{\mathrm{Reeb}}, Y) = 0$ for every $Y \in TM_t$, and $\omega(X_L, X_{\mathrm{Reeb}}) = 1$. This field restricts on any hypersurface M_t to the usual Reeb field for the contact form $\widehat\alpha|_{TM_t}$.

Below we will show that the "height" function $h : (-\varepsilon, 0] \times M \to \mathbb{R}, (t, p) \mapsto t$ is plurisubharmonic with respect to certain almost complex structures.

In the context of this article we will use the term "adapted almost complex structure" in the following sense.

Definition. Let (W, ω) be a symplectic filling of a contact manifold (M, α). An almost complex structure J is a smooth section of the endomorphism bundle $\operatorname{End}(TW)$ such that $J^2 = -\mathbf{1}$. We say that J is **adapted to the filling**, if it is compatible with ω in the usual sense, which means that for all $X, Y \in T_pW$

$$\omega(JX, JY) = \omega(X, Y)$$

holds, and

$$g(X, Y) := \omega(X, JY)$$

defines a Riemannian metric. Additionally, we require J to satisfy close to the boundary $M = \partial W$ the following properties: For the two vector fields X_L and X_{Reeb} introduced above, J is defined as

$$JX_L = X_{\mathrm{Reeb}} \text{ and } JX_{\mathrm{Reeb}} = -X_L ,$$

and J leaves the subbundle $\xi_t = TM_t \cap \ker \widehat{\alpha}$ invariant.

Proposition 1. *Let V be an open subset of $\mathbb{C}$, and let $u : V \to W$ be a J–holomorphic map. The function $h \circ u : V \to \mathbb{R}$ is subharmonic.*

Proof. A short computation shows that $\widehat{\alpha} = -dh \circ J$, and then we get

$$\begin{aligned} 0 \le u^*\omega = u^* d\iota_{X_L}\omega = u^* d\widehat{\alpha} = u^* d\big(-dh \circ J\big) = -u^* dd^c h \\ = -dd^c(h \circ u) = \Big(\frac{\partial^2 h \circ u}{\partial x^2} + \frac{\partial^2 h \circ u}{\partial y^2}\Big)\, dx \wedge dy . \end{aligned}$$ □

Corollary 2. *By the strong maximum principle and the boundary point lemma (e.g.* [GT01]*), any J–holomorphic curve $u : (\Sigma, \partial\Sigma) \to (W, \partial W)$ is either constant or it touches $M = \partial W$ only at its boundary $\partial\Sigma$, and this intersection is transverse. Furthermore, if u is non constant, then the boundary curve $u|_{\partial\Sigma}$ has to intersect the contact structure ξ on ∂W in positive Reeb direction.*

In the rest of the article, we will denote the half space $\big\{z \in \mathbb{C} \bigm| \operatorname{Im} z \ge 0\big\}$ by $\mathbb{H}$. Let $\varphi : N \looparrowright M$ be an immersion of a manifold N in M. We define the self-intersection set of N as

$$N_{\between} := \big\{p \in N \bigm| \exists p' \ne p \text{ with } \varphi(p) = \varphi(p')\big\} .$$

2.2. **Tubular neighborhood theorem for contact submanifolds.** Let N be a contact submanifold of (M, α). The contact structure $\xi = \ker \alpha$ can be split along N into the two subbundles

$$\xi|_N = \xi_N \oplus \xi_N^\perp ,$$

where ξ_N is the contact structure on N given by $\xi_N = TN \cap \xi|_N = \ker \alpha|_{TN}$, and $\xi_N^\perp$ is the symplectic orthogonal of ξ_N inside $\xi|_N$ with respect to the form $d\alpha$. Note that $\xi_N^\perp$ carries a conformal symplectic structure given by $d\alpha$, but neither $\xi_N^\perp$ nor the conformal symplectic structure do depend on the contact form chosen on M. The bundle $\xi_N^\perp$ can be identified with the normal bundle of N.

A well known neighborhood theorem states that $\xi_N^\perp$ determines a small neighborhood of N completely.

Theorem 3. *Let (N, ξ_N) be a contact submanifold of both (M_1, ξ_1) and (M_2, ξ_2). Assume that the two normal bundles $(\xi_1)_N^\perp$ and $(\xi_2)_N^\perp$ are isomorphic as conformal symplectic vector bundles. Then there exists a small neighborhood of N in M_1 that is contactomorphic to a small neighborhood of N in M_2.*

If N has a trivial conformal symplectic normal bundle $\xi_N^\perp$, then we call the product space $N \times \mathbb{R}^{2k}$ with contact structure $\alpha_N + \sum_{j=1}^k (x_j\, dy_j - y_j\, dx_j)$ the *standard model for neighborhoods of N*.

3. The generalized plastikstufe (GPS)

Definition. Let (M,α) be a $(2n+1)$–dimensional contact manifold, and let S be a closed $(n-1)$–dimensional manifold. A **generalized plastikstufe** (**GPS**) is an immersion

$$\Phi : S\times\mathbb{D} \looparrowright M,\ \bigl(s,re^{i\varphi}\bigr)\to\Phi\bigl(s,re^{i\varphi}\bigr)\ ,$$

such that the pull-back $\Phi^*\alpha$ reduces to the form $f(r)\,d\varphi$ with $f\geq 0$ that only vanishes for $r=0$, and $r=1$. Furthermore there is an $\varepsilon>0$ such that self-intersections may only happen between points of the form $(s,re^{i\varphi})$, and $(s',r'e^{i\varphi})$ with $r,r'\in(\varepsilon,1-\varepsilon)$ that have equal φ–coordinate. Finally there must be an open set joining $S\times\{0\}$ with $S\times\partial\mathbb{D}$ that does not contain any self-intersection points.

We call $S\times\{0\}$ (or also its image) the **core** of the GPS, and $S\times\partial\mathbb{D}$ (or again the image) its **boundary**. We denote $S\times\bigl(\mathbb{D}-\{0\}-\partial\mathbb{D}\bigr)$ by GPS^* and call it the **interior** of the GPS.

Remark 2. The regular leaves of the GPS are given by the sets $\{\varphi=\mathrm{const}\}$. We are hence requiring that self-intersections only happen between points lying on the same leaf. A different way to state this requirement consists in saying that there is a continuous map

$$\vartheta : \Phi\bigl(\mathrm{GPS}^*\bigr)\to\mathbb{S}^1$$

such that $\vartheta\bigl(\Phi(s,r,\varphi)\bigr)=\varphi$.

Theorem 4. *A closed contact manifold (M,α) that contains a GPS does not have an exact symplectic filling.*

Remark 3. Using a more precise analysis of bubbling (as in [IS02]) should make it possible to prove that a GPS is an obstruction to finding even a (semipositive) strong symplectic filling. In Remark 4, we sketch how the proof would have to be modified. Note though that [IS02] requires that the self-intersections of the GPS are clean.

4. Constructing immersed plastikstufes in neighborhoods of submanifolds

4.1. **Local construction in codimension two.** The most prominent example of an overtwisted contact manifold in the literature is $\mathbb{R}^3$ with the structure induced by the contact form

$$\alpha_- = \cos r\,dz + r\sin r\,d\varphi\ ,$$

written in cylindrical coordinates (r,φ,z) such that $x=r\cos\varphi$, $y=r\sin\varphi$, and $z=z$. Any plane $\{z=\mathrm{const.}\}$ contains an overtwisted disk centered at the origin with radius $r=\pi$. From the classification in [Eli93], it follows that $(\mathbb{R}^3,\alpha_-)$ is up to contactomorphism the unique contact structure on $\mathbb{R}^3$ that is overtwisted at infinity, and hence any sufficiently small contractible neighborhood of an overtwisted disk in a contact 3–manifold is contactomorphic to $(\mathbb{R}^3,\alpha_-)$.

The Reeb field X_{Reeb} associated to α_- is given by

$$X_{\mathrm{Reeb}} = \frac{1}{r+\sin r\,\cos r}\bigl(\sin r\,\partial_\varphi + (\sin r + r\cos r)\,\partial_z\bigr)\ .$$

Its flow Φ_t is linear, because r remains constant, and the coefficients of the z– and the φ–coordinate only depend on the r–coordinate. The Reeb field is tangent to the overtwisted disk on the circle of radius r_0 such that $r_0=-\tan r_0$ ($r_0\approx 2.029$). Inside this circle X_{Reeb} has a positive z–component, outside it has a negative one. This means the overtwisted disk $\mathbb{D}_{\mathrm{OT}}$ and its translation by the Reeb flow $\Phi_t(\mathbb{D}_{\mathrm{OT}})$ for a time $t\neq 0$ only intersect along the circle of radius r_0 (see Fig. 1). More precisely, the Reeb field reduces on the circle of radius r_0 to $X_{\mathrm{Reeb}}=1/r_0\sin r_0\,\partial_\varphi$, so that it defines a rotation with period $T=2\pi r_0\sin r_0\approx 11.4$.

Take now the product of $\mathbb{R}^3$ with $\mathbb{R}^2$, and define on $\mathbb{R}^3\times\mathbb{R}^2$ the contact form $\alpha_- + R^2\,d\vartheta$, where (R,ϑ) are polar coordinates of $\mathbb{R}^2$. This is a contact fibration, and we will use the first step of the construction in [Pre07], namely we will trace a closed path $\gamma:\mathbb{S}^1\to\mathbb{R}^2$ that has the shape of a figure-eight, with the double point at the origin, and such that both parts of the eight have equal area with respect to the standard area form $2R\,dR\wedge d\vartheta$. Start at the origin of the disk, at $\gamma(1)=0$ on this closed loop, and regard the overtwisted disk $\mathbb{D}_{\mathrm{OT}}$ in the fiber $\mathbb{R}^3\times\{0\}$ described above. By using the parallel transport of $\mathbb{D}_{\mathrm{OT}}$ along the path γ, we are able to describe

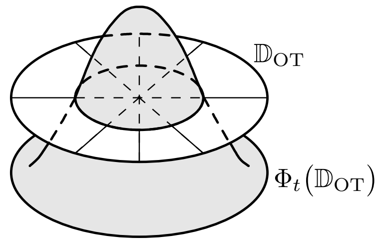

FIGURE 1. The overtwisted disk and its image under the Reeb flow only intersect along a circle of radius r_0.

an immersed plastikstufe. The parallel transport reduces in the fibers to the flow of the vector field $-c\,X_{\mathrm{Reeb}}$ with $c = \|\gamma\|^2\, d\vartheta(\gamma')$, so that the monodromy of a closed loop is just given by the Reeb flow Φ_T for a time T that is equal to the area that has been enclosed by the loop, where we have to count with orientation. The total area of the figure-eight γ vanishes, because on one part of the eight, we are turning in positive direction, on the other in the opposite one, and the area of both parts was chosen to be equal.

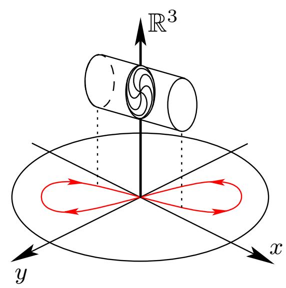

FIGURE 2. Parallel transport of the overtwisted disk along a figure-eight path yields an immersed plastikstufe.

We will describe the construction more explicitely to better understand the self-intersection set. The parallel transport of the overtwisted disk defines an immersion

$$\mathbb{D}_{\mathrm{OT}} \times \mathbb{S}^1 \to \mathbb{R}^3 \times \mathbb{R}^2,\ \big((x, y, 0), e^{i\vartheta}\big) \mapsto \big(\Phi_{T(\vartheta)}(x, y, 0), \gamma(e^{i\vartheta})\big)\ ,$$

where $T(\vartheta) = \int_\gamma r^2\, d\varphi$. The map is well defined, because $T(\vartheta + 2\pi) = T(\vartheta)$. It is also easy to see that this map is an immersion.

The only self-intersection points may lie over the crossing $\gamma(1) = \gamma(-1)$ in the figure-eight, and in fact, since the Reeb flow moves the interior of the overtwisted disk up, and the outer part down, self-intersections only happen between the two circles

$$\big\{(r_0\, \cos\varphi, r_0\, \sin\varphi, 0)\big\} \times \{-1, 1\} \subset \mathbb{D}_{\mathrm{OT}} \times \mathbb{S}^1\ .$$

Denote the area enclosed by one of the petals of the figure-eight path by A. The images of any pair of points $\big((r_0\, \cos\varphi, r_0\, \sin\varphi, 0), 1\big)$ and $\big((r_0\, \cos(\varphi - t_0), r_0\, \sin(\varphi - t_0), 0), -1\big)$, with $t_0 = A/r_0 \sin r_0$ are identical.

Note that if γ is chosen such that $A = 2\pi r_0 \sin r_0$, then the pair of points that intersect each other always lie on the same ray of the overtwisted disk, and we have in fact constructed a GPS. The figure-eight path has to enclose a sufficiently large area, and we cannot realize such a path γ in a disk $\mathbb{D}_R$ of radius $R < 2\sqrt{r_0 \sin r_0} \approx 3.82$.

4.2. **Higher codimension.** Use now the same contact structure on $(\mathbb{R}^3, \alpha_-)$ as above, and extend it to a contact structure on $\mathbb{R}^3 \times \mathbb{C}^k$ with contact form

$$\alpha_- + \sum_{j=1}^k r_j^2\, d\varphi_j\ ,$$

where (r_j, φ_j) are polar coordinates for the j–th $\mathbb{C}$–factor in $\mathbb{C}^k$.

Now we will take the k–fold product of figure-eight loops of different sizes, and group them into a map

$$\Gamma : \mathbb{T}^k \looparrowright \mathbb{C}^k,\ (e^{i\vartheta_1}, \dots, e^{i\vartheta_k}) \mapsto \big(\gamma(e^{i\vartheta_1}), 2^{1/2}\,\gamma(e^{i\vartheta_2}), \dots, 2^{(k-1)/2}\,\gamma(e^{i\vartheta_k})\big)\ .$$

This map is an immersion with self-intersection set

$$\Gamma_{\between} = \big\{(e^{i\vartheta_1}, \dots, e^{i\vartheta_k}) \in \mathbb{T}^k\ \big|\ \text{at least one of the } \vartheta_j \text{ lies in } \pi\mathbb{Z}\big\}\ .$$

Define functions $T_j(\vartheta) := 2^{j-1}\int_0^{\vartheta}\gamma^*\big(r^2\,d\varphi\big)$, and $T(e^{i\vartheta_1}, \dots, e^{i\vartheta_k}) = \sum_{j=1}^k T_j(\vartheta_j)$. Then the immersion

$$\mathbb{D}_{\mathrm{OT}} \times \mathbb{T}^k \to \mathbb{R}^3 \times \mathbb{C}^k,\ \big((x,y,0); e^{i\vartheta_1}, \dots, e^{i\vartheta_k}\big) \mapsto \big(\Phi_{T(\vartheta_1,\dots,\vartheta_k)}(x,y,0); \Gamma(e^{i\vartheta_1}, \dots, e^{i\vartheta_k})\big)\ ,$$

where Φ_t denotes the Reeb flow, is a GPS. Obviously the self-intersection points of this map are contained in the preimage of the self-intersection set $\Gamma_{\between}$ downstairs. Consider two points $(e^{i\vartheta_1}, \dots, e^{i\vartheta_k})$ and $(e^{i\psi_1}, \dots, e^{i\psi_k})$ that have the same image under Γ. It follows for each pair (ϑ_j, ψ_j) that either $\vartheta_j = \psi_j$ or that $\vartheta_j, \psi_j \in \pi\mathbb{Z}$. The disks lying over such points are given by $\Phi_{T(\vartheta)}(\mathbb{D}_{\mathrm{OT}})$ and $\Phi_{T(\psi)}(\mathbb{D}_{\mathrm{OT}})$ respectively, where $\mathbb{D}_{\mathrm{OT}} = \big\{(x,y,0)\ \big|\ \|(x,y,0)\| \le \pi\big\}$. The Reeb flow is φ–invariant and preserves the distance of the points $(x,y,0)$ from the z–axis. Hence in the interior and the exterior of the circle of radius r_0, $\Phi_{T(\vartheta)}(x,y,0)$ can only be equal to $\Phi_{T(\psi)}(x',y',0)$, if $T(\vartheta) = T(\psi)$, because the flow changes the z–coordinate, and by the coefficients chosen in Γ for the paths, T is injective on $(\pi a_1, \dots, \pi a_k)$ with all $a_j \in \{0,1\}$. Additionally then we have $(x,y,0) = (x',y',0)$, so that no self-intersections can happen for points with $\|(x,y,0)\| \ne r_0$. Self-intersections of the GPS can hence only exist for points where the distance of (x,y) from the origin is equal to r_0, but by the size condition on the figure-eight loops the holonomy will always correspond to a rotation by a multiple of 2π so that all conditions of a GPS are satisfied by this map.

4.3. **Application to contact submanifolds.** Let (N, α_N) be an overtwisted contact 3–manifold. We will show that the product manifold $\big(N \times \mathbb{C}^k, \alpha_N + \sum_{j=1}^k r_j^2\,d\vartheta_j\big)$, where (r_j, ϑ_j) are polar coordinates on the j–th factor of $\mathbb{C}^k$ contains a GPS.

Consider a small contractible neighborhood of an overtwisted disk $\mathbb{D}_{\mathrm{OT}}$ in N. This neighborhood is contactomorphic to $(\mathbb{R}^3, \alpha_-)$, because it is overtwisted at infinity. Choose a large ball B in $\mathbb{R}^3$ (so large that the Reeb flow for α_- restricted to the overtwisted disk exists for long enough times), then there is a function $f : N \to \mathbb{R}$ such that the chosen ball B can be embedded by a strict contactomorphism (that means preserving the contact form) into $(N, f\,\alpha_N)$. The contact form $f\,\alpha_N + \sum_{j=1}^k f\,r_j^2\,d\vartheta_j$ on the product manifold $N \times \mathbb{C}^k$ can be transformed by the map $(p; z_1, \dots, z_k) \mapsto (p; z_1/\sqrt{f}, \dots, z_k/\sqrt{f})$ into

$$\big(N \times \mathbb{C}^k, f\,\alpha_N + \sum_{j=1}^k r_j^2\,d\vartheta_j\big)\ .$$

This contains a subset of the form $\big(B \times \mathbb{C}^k, \alpha_- + \sum_{j=1}^k r_j^2\,d\vartheta_j\big)$ in which we can perform the contruction explained above.

Corollary 5. *Let (M, α) be a closed contact $(2n+1)$–manifold that contains an overtwisted contact submanifold N of dimension 3 that has trivial contact normal bundle. There is a neighborhood of N that is contactomorphic to a neighborhood U of $N \times \{0\}$ in the product space $\big(N \times \mathbb{C}^k, \alpha_N + \sum_{j=1}^k r_j^2\,d\vartheta_j\big)$.*

If the neighborhood U contains a sufficiently large disk bundle of $N \times \{0\}$, then it follows that M does not admit an exact symplectic filling.

Proof. By the construction just described $\big(N \times \mathbb{C}^k, \alpha_N + \sum_{j=1}^k r_j^2\,d\vartheta_j\big)$ contains a GPS. Since the GPS is compact, it is contained in some disk bundle around $N \times \{0\}$. If the neighborhood of N is contactomorphic to this disk bundle, then (M, α) contains a GPS, and hence cannot have an exact symplectic filling. ∎

5. Proof of Theorem 4

5.1. **Sketch of the proof.** The proof is based on [Nie06] (which in turn is ultimately based on [Eli88, Gro85]), and it is very helpful to have a good understanding of this first article. Assume that (M,α) has an exact symplectic filling (W,ω). We choose an adapted almost complex structure J on W that has in a neighborhood of the core $S\times\{0\}$ the special form described in [Nie06, Section 3], and in a neighborhood of the boundary $S\times\partial\mathbb{D}$ the particular form described in Section 5.3 below.

The chosen complex structure allows us to write down explicitely the members of a Bishop family around the core of the GPS, so that we find a non-empty moduli space $\mathcal{M}$ of holomorphic disks $u:(\mathbb{D},\partial\mathbb{D})\to(M,\mathrm{GPS}^*)$ with a marked point $z_0\in\partial\mathbb{D}$. The boundary of each holomorphic disk u intersects every regular leaf of the GPS exactly once, or expressed differently the following composition defines a diffeomorphism $\vartheta\circ u|_{\partial\mathbb{D}}:\mathbb{S}^1\to\mathbb{S}^1$ on the circle. The Bishop family is canonically diffeomorphic to a neighborhood of the core $S\times\{0\}$ via the evaluation map

$$\mathrm{ev}_{z_0}:\mathcal{M}\to\mathrm{GPS}^*,\ u\mapsto u(z_0)\ .$$

We can now apply similar intersection arguments for the boundary $S\times\partial\mathbb{D}$ of the GPS (Section 5.3), and for the core ([Nie06, Section 3]), showing that there exists a neighborhood of $\partial\mathrm{GPS}$ that cannot be penetrated by any holomorphic disk, and that the only disks that come close to the core are the ones lying in the Bishop family.

Choose now a smooth generic path $\gamma\subset S\times\mathbb{D}$ that avoids the self-intersection points of the GPS, and that connects the core $S\times\{0\}$ with the boundary of the GPS. In Section 5.2, we define the moduli space $\mathcal{M}_\gamma:=\mathrm{ev}_{z_0}^{-1}(\gamma)$, and show that it is a smooth 1–dimensional manifold. From now on, we will further restrict $\mathcal{M}_\gamma$ to the connected component of the moduli space that contains the Bishop family. Then in fact $\mathcal{M}_\gamma$ has to be diffeomorphic to an open interval. The compactification of one of the ends of the interval simply consists in decreasing the size of the disks in the Bishop family until they collapse to a single point at $\gamma(0)$ on the core of the GPS.

Our aim will be to understand the possible limits at the other end of the interval $\mathcal{M}_\gamma$, and to deduce a contradiction to the fillability of M. The energy of all disks $u\in\mathcal{M}_\gamma$ is bounded by $2\pi\max f$, where $\alpha=f(r)\,d\varphi$ on the GPS. By requiring that the GPS has only clean intersections, we could apply the compactness theorem in [IS02] to deduce even a contradiction for the existence of a semipositive filling (see Remark 4). Instead of merely referring to that result, we have decided to give a full proof of compactness in our situation (see Section 5.4). This way we can drop the stringent conditions on the self-intersections of the GPS, and the required proof is in fact significantly simpler than the full proof of the compactness theorem.

It then follows that for any sequence of disks $u_k\in\mathcal{M}_\gamma$, we find a family of reparametrizations $\varphi_k:\mathbb{D}\to\mathbb{D}$ such that $u_k\circ\varphi_k$ contains a subsequence converging uniformly with all derivatives to a disk $u_\infty\in\mathcal{M}_\gamma$. This means that $\mathcal{M}_\gamma$ is compact, but since at the same time we know that the far-most right element u_∞ in $\mathcal{M}_\gamma$ has a small neighborhood in $\mathcal{M}_\gamma$ homeomorphic to an open interval, it follows that u_∞ is not a boundary point of $\mathcal{M}_\gamma$. Compactness contradicts thus the existence of the filling.

Remark 4. We will briefly sketch how [IS02] could be used to prove the non-existence of even a semipositive filling, if the GPS is cleanly immersed.

The limit of a sequence of holomorphic disks can be described as the union of finitely many holomorphic spheres $u_S^1,\dots,u_S^K$ and finitely many holomorphic disks $v_1,\dots,v_N$. The holomorphic disks $v_j:(\mathbb{D},\partial\mathbb{D})\to(W,\mathrm{GPS})$ are everywhere smooth with the possible exception of boundary points that lie on self-intersections of the GPS. Here v_j will still be continuous though (As an example of such disks, take a figure-eight path in the complex plane $\mathbb{C}$. By the Riemann mapping theorem, there is a holomorphic disk enclosed into each of the loops, but obviously these disks cannot be smooth on their boundary at the self-intersection point of the eight).

We will now first prove that the limit curve of a sequence in $\mathcal{M}_\gamma$ is only composed of a single disk, which then necessarily has to be smooth. Assume we would have a decomposition into several disks $v_1,\dots,v_N$. The boundary of each of these disks v_j is a continuous path in GPS^*, we can hence combine the disks with the projection ϑ defined in Remark 2 to obtain a continuous map $\vartheta\circ v_j|_{\partial\mathbb{D}}:\mathbb{S}^1\to\mathbb{S}^1$. Thus we can associate to each of the disks v_j a degree. In fact it follows

that $\deg \vartheta \circ v_j|_{\partial\mathbb{D}} > 0$, because almost all points on the boundary of v_j are smooth, and for them v_j has to intersect, by Corollary 2, all leaves of the foliation of the GPS in positive direction. Finally assume that there are still several disks, each one necessarily with $\deg \vartheta \circ v_j|_{\partial\mathbb{D}} \geq 1$. This means that the composition of the maps $\vartheta \circ v_j|_{\partial\mathbb{D}}$ will cover the circle several times, but this is not possible for the limit of injective maps $\vartheta \circ u_k|_{\partial\mathbb{D}}$. There is hence only a single disk in the limit. Using Theorem 9 below it finally also follows that this disk is smooth, and has a boundary that lifts to a smooth loop in $S \times \mathbb{D}$.

The reason why there are no holomorphic spheres as bubbles is a genericity argument, since the disk and all spheres are regular smooth holomorphic objects, we can compute the dimension of the bubble tree in which our limit object would lie. By the assumption of semi-positivity, it follows that the dimension would be negative.

5.2. **The moduli space.** The aim of this section is to define the moduli space of holomorphic disks and to prove that it is a smooth manifold. Care has to be taken, because the boundary condition considered in this article is not a properly embedded, but only an immersed submanifold. The main idea is to restrict to those holomorphic curves whose boundary lies locally always on a single leaf of the immersed submanifold. We can then easily adapt standard results.

Let (W, J) be an almost complex manifold, and let L be a compact manifold with $2 \dim L = \dim W$.

Definition. An **immersed totally real submanifold** is an immersion $\varphi : L \looparrowright W$ such that

$$\bigl(D\varphi \cdot T_x L\bigr) \oplus \bigl(J \cdot D\varphi \cdot T_x L\bigr) = T_{\varphi(x)} W$$

at every $x \in L$.

Let $\varphi : L \looparrowright W$ be a totally real immersed submanifold with self-intersection set $L_{\between}$. Choose a (not necessarily connected) submanifold $A \hookrightarrow L$ that is disjoint from $L_{\between}$. Let Σ be a Riemann surface with N boundary components $\partial\Sigma_1, \dots, \partial\Sigma_N$, and choose on each boundary component a marked point $z_j \in \partial\Sigma_j$. Then define $\mathcal{B}(\Sigma; L; A)$ to be the set of maps

$$u : (\Sigma, \partial\Sigma) \mapsto \bigl(W, \varphi(L)\bigr)$$

for which the boundary circles $u|_{\partial\Sigma}$ can be lifted to continuous loops $c : \partial\Sigma \to L$ such that $\varphi \circ c = u|_{\partial\Sigma}$, and $c(z_j) \in A$.

Note that with our conditions the lift of the boundary circles $u|_{\partial\Sigma}$ is unique, because if there were two different loops $c, c' : \partial\Sigma_j \to L$ with $\varphi \circ c = \varphi \circ c'$, and $c(z_j), c'(z_j) \in A$, it follows that the set $\bigl\{z \in \partial\Sigma_j \bigm| c(z) = c'(z)\bigr\}$ contains the point z_j, and is hence non-empty. Furthermore this set is closed, because it is the preimage of the diagonal $\triangle_W := \{(p,p) | \, p \in W\}$ under the map $c \times c' : \partial\Sigma_j \times \partial\Sigma_j \to W \times W$ intersected with the diagonal $\triangle_{\partial\Sigma_j}$. Finally, L can be covered by open sets on each of which the immersion φ is injective, and hence if $c(z) = c'(z)$ there is also an open neighborhood of z on which both paths coincide. It follows that c and c' are equal.

We have to prove that $\mathcal{B}(\Sigma; L; A)$ is a Banach manifold by finding a suitable atlas. To define a chart around a map $u_0 \in \mathcal{B}(\Sigma; L; A)$, construct first a Banach space B_{u_0} by considering the space of sections in $E := u_0^{-1} TW$ satisfying the following boundary condition: Choose the unique collection of loops c that satisfy $\Phi \circ c = u_0|_{\partial\Sigma}$. We can define a subbundle $F \leq E|_{\partial\Sigma}$ over the boundary of the surface by pushing $T_{c(z)} L$ with $D\varphi$ into E. We require the sections $\sigma : \Sigma \to E$ to lie along the boundary $\partial\Sigma$ in the subbundle F, and to be at the marked points $z_j \in \partial\Sigma_j$ tangent to A.

Our aim will be to map these sections in a suitable way into $\mathcal{B}(\Sigma; L; A)$. For this, we first choose a Riemannian metric on L for which A is totally geodesic. Then we extend it to a product metric on $\partial\Sigma \times L$. There is an $\varepsilon_1 > 0$ such that φ restricted to any ε_1–disk centered at an $c(z)$ is an embedding. Furthermore, we find an $\varepsilon_2 > 0$ such that $d\bigl(c(z), c(z')\bigr) < \varepsilon_1/3$ for any pair of points $z, z' \in \partial\Sigma$ such that $d(z, z') < \varepsilon_2$. Let ε be smaller than $\min\{\varepsilon_1/3, \varepsilon_2\}$, and let $U_\varepsilon(c)$ be the ε–neighborhood of the loops $\bigl\{\bigl(z, c(z)\bigr)\bigr\} \subset \partial\Sigma \times L$, i.e., the collection of all points (z', x') that lie at most at distance ε from the set of loops. The restriction of the immersion

$$\mathrm{id} \times \varphi : \partial\Sigma \times L \looparrowright \Sigma \times W, \; (z, x) \mapsto \bigl(z, \varphi(x)\bigr)$$

to $U_\varepsilon(c)$ defines an embedded submanifold of $\Sigma\times W$, because any two points $(z,x),(z',x')\in U_\varepsilon(c)$ for which $\big(z,\varphi(x)\big)=\big(z',\varphi(x')\big)$, obviously satisfy $z=z'$, and as we will show x and x' both lie in an ε_1–disk around $c(z)$ such that $x=x'$. Let $\big(z_0,c(z_0)\big)$ be a point for which $d\big(\big(z_0,c(z_0)\big),(z,x)\big)<\varepsilon$, then using the triangle inequality we get

$$d\big(\big(z,c(z)\big),(z,x)\big)\le d\big(\big(z_0,c(z_0)\big),(z,c(z))\big)+d\big(\big(z_0,c(z_0)\big),(z,x)\big)<\varepsilon_1\ ,$$

which shows that (z,x) lies closer than ε_1 to $(z,c(x))$.

Now we can push the metric from $U_\varepsilon(c)$ forward and extend it to one on $\Sigma\times W$, so that $\big(\mathrm{id}\times\varphi\big)\big(U_\varepsilon(c)\big)$ will be totally geodesic.

Let $\sigma\in B_{u_0}$ be one of the sections of E described above. If σ is sufficiently small, then applying the geodesic exponential map to the section $(0,\sigma)$ in $T(\Sigma\times E)$, and then projecting to the W–component gives a map that lies in the space $\mathcal{B}(\Sigma;L;A)$. The construction described gives a bijection between small sections and maps in $\mathcal{B}(\Sigma;L;A)$ close to u_0. The reason is that there is a smooth map that allows us to regard any manifold in $M_1\times M_2$ tangent to $M_1\times\{x_2\}$ at (x_1,x_2) as a graph over $M_1\times\{x_2\}$ in a neighborhood of that point.

Since we do not see locally the other intersection branches it follows that the Cauchy Riemann equation defines a Fredholm operator on $\mathcal{B}(\Sigma;L;A)$. For a generic adapted almost complex structure J, it follows that the moduli space

$$\widetilde{\mathcal{M}}(\Sigma;L;A)=\big\{u\in\mathcal{B}(\Sigma;L;A)\ \big|\ \bar\partial_J u=0\big\}$$

is a smooth manifold. In our case, we then have that $\mathcal{M}_\gamma:=\widetilde{\mathcal{M}}(\mathbb{D};\mathrm{GPS};\gamma)/G$ is a 1–dimensional manifold.

5.3. **The boundary of the GPS.** The standard definition of the plastikstufe requires the boundary $\partial\mathcal{PS}(S)$ to be a regular leaf of the foliation [Nie06]. That way, $\mathcal{PS}(S)-S\times\{0\}$ is a totally real manifold, and gives thus a Fredholm boundary condition for regarding holomorphic disks, at the same time smooth holomorphic disks in the moduli space have to be transverse to the foliation so that they cannot touch the boundary.

In our definition of the GPS, we want the contact form instead to vanish on the boundary $S\times\partial\mathbb{D}$. In this section, we will show by an intersection argument that there is a neighborhood of the boundary which blocks any holomorphic curve from entering it. Our definition thus implies at this point effectively the same statement as the standard one.

Proposition 6. *Let F be a maximally foliated submanifold inside a contact manifold (M,α). Assume one of its boundary components to be diffeomorphic to $N\cong S\times\mathbb{S}^1$, with S a closed manifold, such that the restriction $\alpha|_{TF}$ of the contact form has the following properties on the collar neighborhood $N\times[0,\varepsilon)=\{(s,e^{i\varphi},r)\}$*

1. *$\alpha|_{TF}$ vanishes on N (in particular N is a Legendrian submanifold), and*
2. *the interior of the collar is foliated and the leaves are $S\times\{e^{i\varphi_0}\}\times(0,\varepsilon)$, for any fixed $e^{i\varphi_0}\in\mathbb{S}^1$.*

Then there is a neighborhood of N in M that is contactomorphic to an open subset of

$$\big(\mathbb{R}\times T^*S\times\mathbb{S}^1\times\mathbb{R},\ dz+\lambda_{\mathrm{can}}-r\,d\varphi\big)$$

such that $N\times[0,\varepsilon)$ lies in this model in $\{0\}\times S\times\mathbb{S}^1\times[0,\varepsilon)$.

Proof. First note that it is clear that the restriction of the contact form can be written on the collar neighborhood as

$$\alpha|_{TF}=f\,d\varphi\ ,$$

with a smooth function $f:\ N\times[0,\varepsilon)\to\mathbb{R}_{\ge0}$ which only vanishes on $N\times\{0\}$. The 2–form $d\alpha$ is a symplectic form on the $(2n)$–dimensional kernel $\xi=\ker\alpha$, so in particular $d\alpha|_{TF}$ cannot vanish on any point $p\in N$, because otherwise T_pF would be an $(n+1)$–dimensional isotropic subspace of $(\xi_p,d\alpha)$. It follows that $\partial_r f(p,0)>0$, and so the map $\Phi:\ N\times[0,\varepsilon)\to N\times\mathbb{R}$, $(p,r)\mapsto(p,f(p,r))$

is after a suitable restriction a diffeomorphism with inverse $\Phi^{-1}(p,r) = (p, f_p^{-1}(r))$, where we wrote $f_p(\cdot) := f(p,\cdot)$. The pull-back of $\alpha|_{TF}$ under Φ^{-1} gives

$$\left(\Phi^{-1}\right)^* (f\, d\varphi) = f(p, f_p^{-1}(r))\, d\varphi = r\, d\varphi\ .$$

Thus, we can assume after changing the orientation of $\mathbb{S}^1$ that α is of the form $-r\, d\varphi$ on the collar neighborhood.

Consider now the normal bundle of the submanifold $N \times [0,\varepsilon)$ in M. A trivialization can be obtained by realizing first that the Reeb field X_{Reeb} is transverse to N, because $TF|_N$ lies in the contact structure, so that there is a small neighborhood, where X_{Reeb} is transverse to F. Choose now an almost complex structure J on $\xi = \ker\alpha$ that is compatible with $d\alpha$ such that J leaves the space on N spanned by $\langle \partial_r, \partial_\varphi\rangle$ invariant. The submanifolds $S_{(e^{i\varphi},r)} := S \times \{(e^{i\varphi}, r)\}$, with $(e^{i\varphi}, r)$ fixed, are all tangent to the contact structure, and it follows that $J \cdot TS_{(e^{i\varphi},r)}$ is transverse to F, because if there was an $X \in TS$, such that $JX \in TF$, then

$$0 < d\alpha(X, JX) = -dr \wedge d\varphi(X, JX) = 0\ .$$

With the tubular neighborhood theorem it follows that there is an open set around $N \times [0,\varepsilon)$ diffeomorphic to $\mathbb{R} \times T^*S \times \mathbb{S}^1 \times (-\varepsilon,\varepsilon)$, and the set $N \times [0,\varepsilon)$ lies in $\{0\} \times S \times \mathbb{S}^1 \times [0,\varepsilon)$.

In the final step, we use a version of the Moser trick explained for example in [Gei06, Theorem 2.24] to find a vector field X_t that isotopes the given contact form into the desired one $dz + \lambda_{\mathrm{can}} - r\, d\varphi$. Let α_t, $t \in [0,1]$, be the linear interpolation between both 1–forms. Assume there is an isotopy ψ_t defined around N such that $\psi_t^*\alpha_t = \alpha_0$. The field X_t generating this isotopy satisfies the equation

$$\mathcal{L}_{X_t}\alpha_t + \dot\alpha_t = 0\ .$$

By writing $X_t = H_t\, R_t + Y_t$, where H_t is a smooth function, R_t is the Reeb vector field of α_t, and $Y_t \in \ker\alpha_t$, we obtain plugging then R_t into the equation above

$$dH_t(R_t) = -\dot\alpha_t(R_t)\ .$$

The vector field Y_t is completely determined by H_t, because Y_t satisfies the equations

$$\begin{aligned}\iota_{Y_t}\alpha_t &= 0\ ,\\ \iota_{Y_t}d\alpha_t &= -dH_t - \dot\alpha_t\ ,\end{aligned}$$

hence it suffices to find a suitable function H_t. Consider the 1-parameter family of Reeb fields R_t as a single vector field on the manifold $[0,1] \times \big(\mathbb{R} \times T^*S \times \mathbb{S}^1 \times \mathbb{R}\big)$. Since R_t is transverse to the submanifold $\widetilde{N} := [0,1] \times \big(\{0\} \times T^*S \times \mathbb{S}^1 \times \mathbb{R}\big)$ along $[0,1] \times N \times [0,\varepsilon)$, it is possible to define a solution H_t to $dH_t(R_t) = -\dot\alpha_t(R_t)$, such that $H_t|_{\widetilde{N}} \equiv 0$. In fact, because $\dot\alpha|_{N\times[0,\varepsilon)} = 0$, it follows that $dH_t|_{N\times[0,\varepsilon)} = 0$, and so the vector field $X_t = H_t\, R_t + Y_t$ vanishes on $N \times [0,\varepsilon$. Hence X_t can be integrated on a small neighborhood of the collar $N \times [0,\varepsilon)$, and $N \times [0,\varepsilon)$ is not moved under the flow, which finishes the proof of the proposition. □

We can easily choose a compatible adapted almost complex structure J on the symplectization

$$\Big(W = \mathbb{R} \times (\mathbb{R} \times T^*S \times \mathbb{S}^1 \times \mathbb{R}),\, \omega = d\big(e^t\, (dz + \lambda_{\mathrm{can}} - r\, d\varphi)\big)\Big)\ ,$$

with coordinates $\{(t, z; \mathbf{q}, \mathbf{p}; e^{i\varphi}, r)\}$. Observe that the Reeb field is given by $X_{\mathrm{Reeb}} = e^{-t}\partial_z$, and that the kernel of α is spanned by the vectors $X - \lambda_{\mathrm{can}}(X)\, \partial_z$ for all $X \in T(T^*S)$, $\partial_\varphi + r\, \partial_z$ and ∂_r. Choose a metric g on S, and let J_0 be the $d\lambda_{\mathrm{can}}$–compatible almost complex structure on T^*S constructed in [Nie06, Appendix B].

With this, we can define a J on W by setting $J\partial_t = X_{\mathrm{Reeb}}$, $JX_{\mathrm{Reeb}} = -\partial_t$, $J\,\partial_r = -\partial_\varphi - r\,\partial_z$, $J(\partial_\varphi + r\,\partial_z) = \partial_r$, and $J(X - \lambda_{\mathrm{can}}(X)\,\partial_z) = J_0X - \lambda_{\mathrm{can}}(J_0X)\,\partial_z$. The last two equations can also be written as $J\,\partial_\varphi = \partial_r + re^t\,\partial_t$ and $JX = J_0X - e^t\,\lambda_{\mathrm{can}}(X)\,\partial_t - \lambda_{\mathrm{can}}(J_0X)\,\partial_z$.

As a matrix, the complex structure J takes the form

$$J(t;z;\mathbf{q},\mathbf{p};e^{i\varphi},r) = \begin{pmatrix} 0 & -e^t & -e^t\,\lambda_{\mathrm{can}} & re^t & 0 \\ e^{-t} & 0 & -\lambda_{\mathrm{can}}\circ J_0 & 0 & -r \\ 0 & 0 & J_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

Note that the center row and column represent linear maps from or to $T(T^*S)$. A lengthy computation shows that this structure is compatible with ω.

Proposition 7. *The almost complex manifold (W,J) can be mapped with a biholomorphism to*

$$(\mathbb{C}\times T^*S\times\mathbb{C}^*, i\oplus J_0\oplus i) .$$

In this model, the contact manifold M is described by the set

$$M\cong\left\{(x+iy;\mathbf{q},\mathbf{p};w)\in\mathbb{C}\times T^*S\times\mathbb{C}^*\;\middle|\; x=-\frac{2+\|\mathbf{p}\|^2+(\ln|w|)^2}{2}\right\} ,$$

and the maximally foliated submanifold F is

$$F\cong\left\{(x;\mathbf{q},0;w)\in\mathbb{R}\times S\times\mathbb{C}^*\;\middle|\; x=-\frac{2+(\ln|w|)^2}{2},\ |w|\geq 1\right\}\subset\mathbb{C}\times T^*S\times\mathbb{C}^* .$$

Proof. The desired biholomorphism is

$$\Phi(t,z;\mathbf{q},\mathbf{p};e^{i\varphi},r) = (\tilde t,\tilde z;\tilde{\mathbf{q}},\tilde{\mathbf{p}};\tilde r e^{i\tilde\varphi}) = \left(-e^{-t}-F-\frac{r^2}{2}, z;\mathbf{q},\mathbf{p};e^re^{i\varphi}\right) ,$$

with the function

$$F:\,T^*M\to\mathbb{R},\ (\mathbf{q},\mathbf{p})\mapsto\frac{\|\mathbf{p}\|^2}{2} .$$

It brings J into standard form with respect to the coordinate pairs $(\tilde r e^{i\tilde\varphi})$, $(\tilde t,\tilde z)$. More explicitely, by pulling back J under the diffeomorphism

$$\Phi^{-1}(\tilde t,\tilde z;\tilde{\mathbf{q}},\tilde{\mathbf{p}};\tilde r e^{i\tilde\varphi}) = (t,z;\mathbf{q},\mathbf{p};e^{i\varphi},r) = \left(-\ln(-\tilde t-F-\frac{(\ln\tilde r)^2}{2}),\tilde z;\tilde{\mathbf{q}},\tilde{\mathbf{p}};e^{i\tilde\varphi},\ln\tilde r\right) ,$$

i.e. by computing $D\Phi\cdot J\cdot D\Phi^{-1}$, we obtain the matrix

$$D\Phi\cdot J\cdot D\Phi^{-1} = \begin{pmatrix} 0 & -1 & -\lambda_{\mathrm{can}}-dF\circ J_0 & 0 & 0 \\ 1 & 0 & dF-\lambda_{\mathrm{can}}\circ J_0 & 0 & 0 \\ 0 & 0 & J_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\tilde r \\ 0 & 0 & 0 & \tilde r & 0 \end{pmatrix} ,$$

and since, according to [Nie06, Appendix B], $dF\circ J_0 = -\lambda_{\mathrm{can}}$, this gives the desired normal form. □

Proposition 8. *Let F be a maximally foliated submanifold in a contact manifold (M,α). Let (W,ω) be a symplectic filling of M, and assume F to have a boundary component of the type explained in Proposition 6. There is a neighborhood U of the boundary with an almost complex structure, which prevents any holomorphic curve $u:\,(\Sigma,\partial\Sigma)\to(W,F)$ that has in F contractible boundary components, from entering U.*

Proof. Choose for the neighborhood U of ∂F the model described in Proposition 7 together with the almost complex structure J given there. This J can be easily extended over the whole filling (W,ω). Note that the neighborhood is foliated by J-holomorphic codimension 2 manifolds of the form $N_C := \{C\}\times T^*S\times\mathbb{C}^*$ for any fixed complex number C.

Let now $u:\,(\Sigma,\partial\Sigma)\to(W,F)$ be a holomorphic curve that has in F contractible boundary, and assume that u intersects the model neighborhood U. Write the restriction of u to $V:=u^{-1}(U)\subset\Sigma$ as

$$u|_V:\,V\to\mathbb{C}\times T^*S\times\mathbb{C}^*,\ z\mapsto\big(u_1(z);\mathbf{q}(z),\mathbf{p}(z);u_2(z)\big) .$$

First, we will show that the imaginary part of the first coordinate u_1 is constant. If it was not, then there would be by Sard's Theorem a regular value $c_y \in \mathbb{R}$, such that $u_1^{-1}(\mathbb{R} + ic_y)$ consists of finitely many regular 1–dimensional submanifolds. The real part of u_1 changes along these submanifolds, because u_1 satisfies the Cauchy-Riemann equation. Hence it is possible to find a complex number $c_x + ic_y \in \mathbb{C}$ such that $N_{c_x+ic_y}$ has finitely many transverse intersection points with u. By our assumption, it is possible to cap off the holomorphic curve u by adding disks that lie inside F. Note that $N_{c_x+ic_y}$ is the boundary of the submanifold

$$\widetilde{N}_{c_x+ic_y} := \bigl\{x + ic_y \bigm| x \in [c_x, \infty)\bigr\} \times T^*S \times \mathbb{C}^* .$$

The intersection of $\widetilde{N}_{c_x+ic_y}$ with M gives a submanifold that is disjoint from F, and this submanifold together with $N_{c_x+ic_y}$ represents the trivial homology class in $H_{2n-2}(W)$.

The only intersections between the capped off holomorphic curve and $\partial(W \cap \widetilde{N}_{c_x+ic_y})$ lie in the subsets, where both classes are represented by J-holomorphic manifolds. Hence the intersection number is positive, but since $\partial(W \cap \widetilde{N}_{c_x+ic_y})$ represents the trivial class in homology, this is clearly a contradiction.

It follows that $\operatorname{Im} u_1$ is constant, and with the Cauchy-Riemann equation, we immediately obtain that the real part of u_1 must also be constant. This in turn means that the holomorphic curve is completely contained in the neighborhood, because the only way that u could not be completely contained in the neighborhood U is if $|u_2(z)|$ changes sufficiently or if $\mathbf{p}$ grows, but in both cases u will hit the hypersurface M before leaving U. Hence u is contained in U. Consider now the T^*S–part of u. Since u sits on F, it follows that the T^*S–part has boundary on the zero-section of T^*S, and so it has no energy, and is thus constant. So far, it follows that $u(z)$ can be written as $u(z) = (c_x + ic_y; \mathbf{q}_0, 0; f(z))$, where $f : \mathbb{D} \to \mathbb{C}^*$ such that $f(\mathbb{S}^1)$ lies in one of the circles of fixed radius R or $1/R$. But in fact, only the circle of radius $R > 1$ lies in F, hence all boundary component of Σ are mapped to the circle of radius R, and so by the maximum principle $|f|^2$ is bounded by R^2, and by the boundary point lemma the derivative of f along the boundary may nowhere vanish. □

5.4. **Bubbling analysis.** To obtain compactness of our moduli space, we need to distinguish two cases: Either the first derivatives of the sequence are uniformly bounded from the beginning, and we have subsequence with a clean limit (after adapting the standard result to the immersed boundary condition), or if the first derivatives explode, we show that we do find a global uniform bound on the derivatives if we reparametrize the disks in a suitable way.

Theorem 9. *Let Σ be a Riemann surface that does not need to be compact, and may or may not have boundary. Let $\Omega_k \subset \Sigma$ be a family of increasing open sets that exhaust Σ, i.e.,*

$$\cup_k \Omega_k = \Sigma \text{ and } \Omega_k \subset \Omega_{k'} \text{ for } k \leq k'.$$

Define $\partial\Omega_k := \Omega_k \cap \partial\Sigma$. Let (W, J) be a compact almost complex manifold that contains a totally real immersion $\varphi : L \looparrowright W$ of a compact manifold L.

Let u_k be a sequence of holomorphic maps $u_k : \bigl(\Omega_k, \partial\Omega_k\bigr) \to \bigl(W, \varphi(L)\bigr)$ whose derivatives are uniformly bounded on compact sets, i.e., if $K \subset \Sigma$ is a compact set, then there exists a constant $C(K) > 0$ such that

$$\|Du_k(z)\| \leq C(K)$$

for all k and all $z \in \Omega_k \cap K$. Additionally assume that the restriction of u_k to the boundary $\partial\Omega_k$ lifts to a collection of smooth paths $u_k^L : \partial\Omega_k \to L$ such that $\varphi \circ u_k^L = u_k|_{\partial\Omega_k}$.

Then there exists a subsequence of u_k that converges on any compact subset uniformly with all derivatives to a holomorphic curve $u_\infty : (\Sigma, \partial\Sigma) \to \bigl(W, \varphi(L)\bigr)$, whose boundary lifts to a collection of smooth paths $u_\infty^L : \partial\Sigma \to L$, and the boundary paths u_k^L also converge locally uniformly to u_∞^L.

Proof. The theorem is well-known in case that $\partial\Sigma = \emptyset$ or that $\varphi(L)$ is an embedded totally real submanifold (see for example [MS04, Theorem 4.1.1]). In fact, our situation can be reduced to one, where we can apply this standard result. Using Arzelà-Ascoli it is easy to find a subsequence u_k that converges uniformly in C^0 on any compact set to a continuous map u_∞, and such that the lifts $u_k^L : \partial\Omega_k \to L$ converge in C^0 on any compact set to a lift $u_\infty^L : \partial\Sigma \to L$ with $\varphi \circ u_\infty^L = u_\infty|_{\partial\Sigma}$.

Let $K \subset \Sigma$ be a compact set on which we want to show uniform C^∞–convergence. If $\partial K := K \cap \partial\Sigma$ is empty, then the uniform converges for the derivatives follows from the standard result. If ∂K is non-empty, then cover $u_\infty^L(\partial K)$ with a finite collection of open sets $V_1, \dots, V_N$ on each of which φ is injective. We can choose smaller open subsets $V_j' \subset V_j$ whose closure $\overline{V_j'}$ is also contained in V_j, and whose union $V_1' \cup \dots \cup V_N'$ still cover $u_\infty^L(\partial K)$.

Cover also K itself with open sets U_k that either do not intersect the boundary ∂K or if $U_k \cap \partial K \neq \emptyset$, then there is a V_j' such that $u_\infty^L(U_k \cap \partial K) \subset V_j'$. Only finitely many U_k are needed to cover K. We get for every $U_k \cap K$ uniform C^∞–convergence, because if U_k intersects now ∂K we can use the standard result: For n large enough $u_n(U_k \cap \partial K \cap \Omega_n)$ will be contained in the larger subset V_j, on which φ is an embedding. □

Theorem 10 (Gromov compactness)**.** *Let*

$$u_n : (\mathbb{D}, \partial\mathbb{D}) \to (W, \mathrm{GPS})$$

be a sequence of holomorphic disks that represent elements in the moduli space $\mathcal{M}_\gamma$.

There exists a family $\varphi_n : \mathbb{D} \to \mathbb{D}$ of biholomorphisms such that $u_n \circ \varphi_n$ contains a subsequence converging uniformly in C^∞ to a holomorphic disk

$$u_\infty : (\mathbb{D}, \partial\mathbb{D}) \to (W, \mathrm{GPS})$$

that represents again an element in $\mathcal{M}_\gamma$.

Proof. Choose an arbitrary J–compatible metric on W, and endow the disk $\mathbb{D} \subset \mathbb{C}$ with the standard metric g_0 on the complex plane. Denote by $\|Du_k(z)\|_{\mathbb{D}}$ the norm of the differential of u_k at a point $z \in \mathbb{D}$ with respect to g_0 on the disk, and the chosen metric on W. If $\|Du_k\|_{\mathbb{D}}$ is uniformly bounded for all $k \in \mathbb{N}$ and all $z \in \mathbb{D}$, then by Theorem 9 we are done.

So assume this to be false, then there exists (by going to a subsequence if necessary) a sequence $z_k \in \mathbb{D}$ such that

$$\|Du_k(z_k)\|_{\mathbb{D}} \to \infty ,$$

and in fact by using rotations, we may assume that all z_k lie on the interval $[0, 1]$.

Let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane $\{z | \operatorname{Im} z \geq 0\}$ endowed with the standard metric, and denote by $\|Dv(z)\|_{\mathbb{H}}$ the norm of the differential of a map $v : \mathbb{H} \to W$ at a point $z \in \mathbb{H}$ with respect to the standard metric on the half plane, and the chosen metric on W. We can map the half plane into the unit disk using the biholomorphism

$$\Phi : \mathbb{H} \to \mathbb{D} - \{-1\},\, z \mapsto \frac{i - z}{i + z} ,$$

and use this to pull-back the sequence of disks to $u_k^{\mathbb{H}} := u_k \circ \Phi : (\mathbb{H}, \mathbb{R}) \to (W, \mathrm{GPS})$. The map Φ is not an isometry, but on compact sets of the upper half plane, $\Phi^* g_0$ is equivalent to the standard metric. Hence it follows that also $\|Du_k^{\mathbb{H}}\|_{\mathbb{H}}$ cannot be bounded on the segment $I := \{it | \, t \in [0, 1]\}$. Let $x_k \in I$ be a point where $\|Du_k^{\mathbb{H}}\|_{\mathbb{H}}$ takes its maximum on I.

Apply the Hofer Lemma (see for example [MS04, Lemma 4.6.4]) for fixed k, and $\delta = 1/2$, that means, restrict $\|Du_k^{\mathbb{H}}\|_{\mathbb{H}}$ to the unit disk $\mathbb{D}(x_k) \cap \mathbb{H}$. There is a positive ε_k with $\varepsilon_k \leq 1/2$, and a $y_k \in \mathbb{D}_{1/2}(x_k) \cap \mathbb{H}$ such that

$$\|Du_k^{\mathbb{H}}(x_k)\|_{\mathbb{H}} \leq 2\varepsilon_k \, \|Du_k^{\mathbb{H}}(y_k)\|_{\mathbb{H}}$$

and

$$\|Du_k^{\mathbb{H}}(z)\|_{\mathbb{H}} \leq 2 \, \|Du_k^{\mathbb{H}}(y_k)\|_{\mathbb{H}}$$

for all $z \in \mathbb{D}_{\varepsilon_k}(y_k) \cap \mathbb{H}$.

Set $c_k := \|Du_k^{\mathbb{H}}(y_k)\|_{\mathbb{H}}$. First we will show that for large k, all the disks $\mathbb{D}_{\varepsilon_k}(y_k)$ intersect the boundary $\partial\mathbb{H} = \mathbb{R}$ of the half plane. Even stricter, there exists a constant $K > 0$ such that $c_k \operatorname{Im}(y_k) < K$ for all k (if the disks intersect the real line, we have $\operatorname{Im} y_k < \varepsilon_k$, multiplying with c_k on both sides would still allow the left side to be unbounded). Suppose that such a constant

did not exist, so that by going to a subsequence, $c_k \operatorname{Im} y_k$ converges monotonously to ∞. Define $H_k := \{z \in \mathbb{C} |\ \operatorname{Im} z \geq -c_k \operatorname{Im} y_k\}$, and a sequence of biholomorphisms

$$\varphi_k : \mathbb{D}_{\varepsilon_k c_k} \cap H_k \to \mathbb{D}_{\varepsilon_k}(y_k) \cap \mathbb{H},\ z \mapsto y_k + \frac{z}{c_k}\ .$$

Pulling back, we find holomorphic maps $\widehat{u}_k := u_k^{\mathbb{H}} \circ \varphi_k : \mathbb{D}_{\varepsilon_k c_k} \cap H_k \to W$ with $\|D\widehat{u}_k(0)\| = 1$, and $\|D\widehat{u}_k\| \leq 2$ everywhere else. Using Theorem 9 (or just for example [MS04, Theorem 4.1.1]), proves that there exists a subsequence that converges locally uniformly with all derivatives to a non-constant map $\widehat{u}_\infty : \mathbb{C} \to W$. The standard removal of singularity theorem yields then a non-constant holomorphic sphere, which cannot exist in an exact symplectic manifold. Thus there is a constant $K > 0$ such that $c_k\ \operatorname{Im}(y_k) < K$.

Now we slightly modify the charts used above to keep the boundary of the reparametrized domains on the height of the real line. Set $y_k' := c_k \operatorname{Im} y_k$ and $r_k := \varepsilon_k c_k$, and consider the following sequence of biholomorphisms

$$\psi_k : \mathbb{D}_{r_k}(iy_k') \cap \mathbb{H} \to \mathbb{D}_{\varepsilon_k}(y_k) \cap \mathbb{H},\ z \mapsto \frac{z}{c_k} + \operatorname{Re} y_k\ .$$

Note that the intersection of $\mathbb{D}_{\varepsilon_k}(y_k)$ with the real line is given by the interval

$$\mathbb{D}_{\varepsilon_k}(y_k) \cap \mathbb{R} = \mathbb{D}(x_k) \cap \mathbb{R} \subset (-1, 1)\ .$$

The image of the interval $(-1, 1)$ under Φ is the segment on the boundary of the unit disk enclosed between the angles $(-\pi/2, \pi/2)$. This means that the boundary part of the disk that is affected by the reparametrization lies on the right half of the complex plane.

On the domain of the reparametrized maps $\widehat{u}_k := u_k^{\mathbb{H}} \circ \psi_k : \mathbb{D}_{r_k}(iy_k') \cap \mathbb{H} \to W$ we have $\|D\widehat{u}_k\| \leq 2$, and $\|D\widehat{u}_k(iy_k')\| = 1$. We can also find a subsequence of $\widehat{u}_k$ with increasing domains, i.e., $\mathbb{D}_{r_k}(iy_k') \subset \mathbb{D}_{r_l}(iy_l')$ for all $l \geq k$, by using that the y_k' are all bounded while the radii of the disks r_k become arbitrarily large. Then Theorem 9 provides a subsequence of the $\widehat{u}_k$ that converges locally uniformly with all derivatives to a holomorphic map $\widehat{u}_\infty : (\mathbb{H}, \mathbb{R}) \to (W, \mathrm{GPS})$. To see that $\widehat{u}_\infty$ is not constant, take a subsequence such that y_k' converges to y_∞'. The norm of the derivative of u_∞ at iy_∞ is $\|D\widehat{u}_\infty(iy_\infty')\| = 1$, because $\|D\widehat{u}_\infty(iy_\infty') - D\widehat{u}_k(iy_k')\| \leq \|D\widehat{u}_\infty(iy_\infty') - D\widehat{u}_\infty(iy_k')\| + \|D\widehat{u}_\infty(iy_k') - D\widehat{u}_k(iy_k')\|$ becomes arbitrarily small. The first term is small, because the differential of $\widehat{u}_\infty$ is continuous, the second can be estimated by using that the convergence of $\widehat{u}_k$ to $\widehat{u}_\infty$ is uniform on a small compact neighborhood of iy_∞'.

Let us come back to the initial family of disks $u_k : (\mathbb{D}, \partial\mathbb{D}) \to (W, \mathrm{GPS})$. The maps ψ_k induce reparametrizations of the whole disk by $\Phi \circ \psi_k \circ \Phi^{-1}$. The image of a compact subset of $\mathbb{D} - \{-1\}$ under Φ^{-1} is a compact subset in $\mathbb{H}$, so that we get on any compact subset of $\mathbb{D} - \{-1\}$ uniform C^∞–convergence of $u_k \circ \psi_k$ to $u_\infty := \widehat{u}_\infty \circ \Phi^{-1}$. To complete the proof of our compactness theorem, we have to show that the first derivatives of $u_k \circ \psi_k$ are also uniformly bounded in a neighborhood of $\{-1\}$.

Rotate the disk $\mathbb{D}$ by multiplying its points by $e^{i\pi} = -1$ such that -1 lies at 1. Then the holomorphic curve

$$\big(\mathbb{H} - \{0\}, (-\infty, 0) \cup (0, \infty)\big) \to (W, \mathrm{GPS}), z \mapsto u_\infty\big(-\Phi(z)\big)$$

has finite energy, and we can apply the removal of singularity theorem in the form described in Theorem 11. The consequence for u_∞ is that the composition $\vartheta \circ u_\infty|_{\partial\mathbb{D} - \{-1\}}$ extends to a continuous map $\mathbb{S}^1 \to \mathbb{S}^1$ that is strictly monotonous. In fact, $\vartheta \circ u_\infty|_{\partial\mathbb{D} - \{-1\}}$ covers the whole circle with exception of the point

$$e^{i\,\varphi_\infty} := \lim_{e^{i\varphi} \to -1} \vartheta \circ u_\infty(e^{i\varphi})\ ,$$

and so for any ε–neighborhood $U_\varepsilon \subset \mathbb{S}^1$ of $e^{i\,\varphi_\infty}$, we find a $\delta > 0$ such that $\{\vartheta \circ u_k \circ \psi_k(e^{i\varphi}) |\ \varphi \in (-\pi/2 + \delta, \pi/2 - \delta)\}$ covers for any sufficiently large k the complement $\mathbb{S}^1 - U_\varepsilon$ of U_ε. Let K be the segment $\{e^{i\varphi} |\ \varphi \in (-\pi/2, \pi/2)\}$. Remember that the images of K exhausts $\partial\mathbb{D} - \{-1\}$ if we apply the reparametrizations ψ_k, and so it follows in particular that the unparametrized disks $u_k : (\mathbb{D}, \partial\mathbb{D}) \to (W, \mathrm{GPS})$ intersect on K for sufficiently large k almost all leaves of the foliation of the GPS.

Assume now that the first derivatives of the $u_k \circ \psi_k$ are not uniformly bounded in a neighborhood of $\{-1\}$. By the same reasoning, it follows that the $u_k \circ \psi_k$ intersect almost all leaves of the GPS on the segment $K' = \{e^{i\varphi} \,|\, \varphi \in (\pi/2, \pi/3)\}$, but this yields a contradiction. $\square$

In our special situation, we only need the following very weak form of removal of singularity, which states that a holomorphic curve that has a puncture on its boundary approaches the same leaf of the foliation from both sides of the puncture.

Theorem 11 (Removal of singularity)**.** *Let (W, ω) be a compact manifold with exact symplectic form $\omega = d\alpha$, and with convex boundary $\partial W = (M, \alpha)$. Assume that M contains a GPS $\varphi : S \times \mathbb{D} \hookrightarrow M$, and choose an adapted almost complex structure J on W. Assume*

$$u : \Big(\mathbb{D}_\varepsilon \cap \mathbb{H} - \{0\}, (-\varepsilon, 0) \cup (0, \varepsilon)\Big) \to \big(W, \mathrm{GPS}\big)$$

to be a non-constant holomorphic curve that has finite energy. Recall that there was a continuous map $\vartheta : \mathrm{GPS}^* \to \mathbb{S}^1$ that labels the leaves of the foliation on the GPS.

We find a continuous path $\widehat{c} : (-\varepsilon, \varepsilon) \to \mathbb{S}^1$ with

$$\widehat{c}|_{(-\varepsilon,0)\cup(0,\varepsilon)} = \vartheta \circ u|_{(-\varepsilon,0)\cup(0,\varepsilon)} \ .$$

A more geometric way to state this result is to say that the boundary segments of the holomorphic curve approach from both sides of 0 *asymptotically the same leaf.*

Proof. One of the basic ingredients in all proofs of this type is the following estimate for the energy of u

$$E(u) = \int_{\mathbb{D}_\varepsilon \cap \mathbb{H} - \{0\}} u^*\omega = \int_0^\varepsilon \int_{\gamma_r} \frac{|\partial_\varphi u|^2}{r^2}\, r\, dr \wedge d\varphi \geq \int_0^\varepsilon \left(\int_{\gamma_r} |\partial_\varphi u|\, d\varphi\right)^2 \frac{dr}{2\pi r} = \int_0^\varepsilon \frac{L(\gamma_r)^2}{2\pi r}\, dr \ ,$$

where γ_r is the image $\big\{u(re^{i\varphi}) \;\big|\; \varphi \in [0, \pi]\big\}$ of the half-circle of radius r in the hyperbolic plane, and $L(\gamma_r)$ is its length with respect to the compatible metric on W. It is clear that $L(\gamma_r)$ cannot be bounded from below, because the energy $E(u)$ is finite.

Denote the segments composing the map $\vartheta \circ u|_{(-\varepsilon,0)\cup(0,\varepsilon)}$ by $c_- : (-\varepsilon, 0) \to \mathbb{S}^1$, and $c_+ : (0, \varepsilon) \to \mathbb{S}^1$. By Corollary 2, both maps $c_\pm$ are strictly increasing.

It easily follows that the $c_\pm$ are bounded close to 0 (in the sense that they do not turn infinitely often as $z \to \infty$), because there is a sequence of radii r_k with $r_k \to 0$ such that $L(\gamma_{r_k}) \to 0$. Denote $(\mathbb{D}_{r_1} - \mathbb{D}_{r_k}) \cap \mathbb{H}$ by $D(r_1, r_k)$. Then

$$E\big(u|_{D(r_1,r_k)}\big) = \int_{\partial D(r_1,r_k)} u^*\alpha \geq \int_{[-r_1,\,-r_k]\,\cup\,[r_k,\,r_1]} u^*\alpha - \big(L(\gamma_{r_1}) + L(\gamma_{r_k})\big)\, \max \|\alpha\| \to \infty \ .$$

It follows that we find continuous extensions $\widehat{c}_- : (-\varepsilon, 0] \to \mathbb{S}^1$, and $\widehat{c}_+ : [0, \varepsilon) \to \mathbb{S}^1$. If $\widehat{c}_-(0) = \widehat{c}_+(0)$, we are done, so assume these limits to be different. Choose a small $\delta > 0$, such that the δ–neighborhoods $U_-, U_+ \subset \mathbb{S}^1$ around $\widehat{c}_-(0)$ and $\widehat{c}_+(0)$ respectively do not overlap. There is an $\varepsilon' > 0$ for which the segment $[0, \varepsilon')$ is contained in $\widehat{c}_+^{-1}(U_+)$, and $(-\varepsilon', 0]$ is contained in $\widehat{c}_-^{-1}(U_-)$, and all the points in $u\big((0, \varepsilon')\big)$ are at distance more than $C > 0$ from the points in $u\big((-\varepsilon', 0)\big)$. In particular it follows that the length $L(\gamma_r)$ for any $r \in (0, \varepsilon')$ is larger than C, and so by the energy inequality at the beginning of the proof, we get a contradiction to $\widehat{c}_-(0) \neq \widehat{c}_+(0)$. $\square$

6. Outlook and open questions

One obvious application of the observations made in this paper is the definition of a capacity invariant for contact manifolds. Unfortunately, we were not able to measure the "size" efficiently in a numerical way so that our invariant is rather rough.

To measure the capacity, we choose a contact manifold (N, ξ_N) that will serve as the "testing probe".

Then we can define for any contact manifold (M,ξ) with $\dim M = 2k + \dim N$, and $k \geq 1$, an invariant C_{ξ_N} defined as follows

$$C_{\xi_N}(M,\xi) = \begin{cases} 0 & (N,\xi_N) \text{ cannot be embedded with trivial normal bundle into } M; \\ \infty & N \times \mathbb{R}^{2k} \text{ with the standard contact form can be embedded into } M; \\ 1 & \text{otherwise, that means } (N,\xi_N) \text{ can be embedded with trivial normal} \\ & \text{bundle into } M\text{, but not with the full neighborhood.} \end{cases}$$

This way, we obtain for the standard sphere $\left(\mathbb{S}^{2n-1},\xi_0\right)$ that $C_{\xi_0}(M,\xi) = \infty$ for any contact manifold (M,ξ). If $\left(N,\xi_-\right)$ is an overtwisted contact 3–manifold, and if (M,ξ) is a manifold with exact symplectic filling, then $C_{\xi_-}(M,\xi) < \infty$.

The most important problem in this context would be to find examples of contact manifolds that do allow the embedding of an overtwisted contact manifold N with the full model neighborhood $N \times \mathbb{R}^{2k}$, because otherwise it is so far unclear whether the capacity C_{ξ_N} is able to distinguish any manifolds. Possible candidates to check are the following:

Question 1. *Let (M,α) be a closed contact manifold. Bourgeois described in* [Bou02] *a construction of a contact structure on $M \times \mathbb{T}^2$ for which every fiber $M \times \{p\}$ with $p \in \mathbb{T}^2$ is contactomorphic to the initial manifold. How large is the tubular neighborhood of such a fiber?*

Question 2. *Giroux conjectures that contact manifolds of arbitrary dimension obtained from the negative stabilization of an open book should be "overtwisted". The simplest example of such a manifold is a sphere $(\mathbb{S}^{2n-1},\alpha_-)$ constructed by taking the cotangent bundle $T^*\mathbb{S}^{n-1}$ for the pages, and a negative Dehn-Seidel twist as the monodromy map (see Example 5). How large is the tubular neighborhood of $(\mathbb{S}^3,\alpha_-)$ in a higher dimensional sphere?*

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